On the prediction error for singular stationary processes

Mamikon S. Ginovyan

Boston University

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The Model

- Let $\{X(t),\ t \in \mathbb{Z}\}$ be a centered real-valued second-order stationary process with covariance function $r(t)$ and spectral measure μ:

$$r(t) = \int_{-\pi}^{\pi} e^{-it\lambda} d\mu(\lambda), \quad t \in \mathbb{Z}. \tag{2.1}$$

- Lebesgue decomposition of μ:

$$d\mu(\lambda) = d\mu_a(\lambda) + d\mu_s(\lambda) = f(\lambda)d\lambda + d\mu_s(\lambda), \tag{2.2}$$

- $f(\lambda)$ is the *spectral density* of $X(t)$.
- We assume that $X(t)$ is *non-degenerate*: $\mathrm{Var}[X(0)] = r(0) > 0$, and the spectral measure μ is *non-trivial*, i.e., μ has infinite support.

The prediction problem

- The *"finite" linear prediction problem* is as follows.
- Suppose we observe a finite realization of the process $X(t)$:

$$\{X(t),\ -n \leq t \leq -1\}, \quad n \in \mathbb{N} := \{1, 2, \ldots\}.$$

- We want to make an one-step ahead prediction, that is, to predict the unobserved random variable $X(0)$, using the *linear predictor*

$$Y = \sum_{k=1}^{n} c_k X(-k).$$

- The coefficients c_k $(k = 1, 2, \ldots, n)$ are chosen so as to minimize the *mean-squared error (MSE)*: $E\,|X(0) - Y|^2$.

The prediction problem

- If $\widehat{c}_k := \widehat{c}_{k,n}$ are the minimizing constants, then the random variable

$$\widehat{X}_n(0) := \sum_{k=1}^{n} \widehat{c}_k X(-k)$$

is called the *best linear one-step ahead predictor* of $X(0)$ based on the observed finite past: $X(-n), \ldots, X(-1)$.

- The minimum MSE:

$$\sigma_n^2(f) := E\left|X(0) - \widehat{X}_n(0)\right|^2 \geq 0$$

is called the *prediction error* of $X(0)$ based on the past of length n.

- Observe that

$$\sigma_{n+1}^2(f) \le \sigma_n^2(f), \quad n \in \mathbb{N},$$

and hence the limit of $\sigma_n^2(f)$ as $n \to \infty$ exists. Denote by

$$\sigma^2(f) := \sigma_\infty^2(f)$$

the prediction error by the entire infinite past: $\{X(t),\ t \le -1\}$.
- From the prediction point of view it is natural to distinguish:
- The class of processes for which we have *error-free prediction* by the entire infinite past, that is, $\sigma^2(f) = 0$. Such processes are called *deterministic* or *singular*,
- Processes for which $\sigma^2(f) > 0$ are called *nondeterministic* or *regular*.

The prediction problem

- Define the *relative prediction error*

$$\delta_n(f) := \sigma_n^2(f) - \sigma^2(f),$$

and observe that

$$\delta_n(f) \geq 0 \quad \text{and} \quad \delta_n(f) \to 0 \quad \text{as} \quad n \to \infty.$$

- But what about the speed of convergence of $\delta_n(f)$ to zero?
- This speed depends on the regularity nature (regular or singular) of the observed process $X(t)$.
- In this talk we discuss this question.

The prediction problem

- Specifically, the prediction problem we are interested in is *to describe the rate of decrease of $\delta_n(f)$ to zero as $n \to \infty$*, depending on the regularity nature of the observed process $X(t)$.
- It turns out that
- for regular processes the asymptotic behavior of $\delta_n(f) = \sigma_n^2(f) - \sigma^2(f)$ is determined by
    - the dependence structure of the observed process $X(t)$ and
    - the differential properties of its spectral density f, while
- for singular processes $(\delta_n(f) = \sigma_n^2(f))$ it is determined by
    - the geometric properties of the spectrum of $X(t)$ and
    - singularities of its spectral density f.
- In this talk we focus on the less investigated case - singular processes.

- This talk is based on the following joint works with Nikolay Babayan (Russian-Armenian University) and Murad Taqqu (Boston University).
  1. N. M. Babayan, M. S. Ginovyan. On asymptotic behavior of the prediction error for a class of deterministic stationary sequences. *Acta Math. Hungar., 167 (2), 501–528 (2022)*..
  2. N. M. Babayan, M. S. Ginovyan. On the prediction error for singular stationary processes and transfinite diameters of related sets. *Zapiski POMI*, v. 510, 28–50 (2022).
  3. N. M. Babayan, M. S. Ginovyan, M. S. Taqqu. Extensions of Rosenblatt's results on the asymptotic behavior of the prediction error for deterministic stationary sequences. *J. Time Ser. Anal.*, 42: 622–652, 2021.
  4. N. M. Babayan, M. S. Ginovyan. Asymptotic behavior of the prediction error for stationary sequences. *Probability Surveys*. 20: 664–721, 2023.

Kolmogorov-Szegő Theorem
Spectral characterization
of singular and regular processes

Kolmogorov-Szegő Theorem

The next result describes the asymptotic behavior of $\sigma_n^2(\mu)$ for a stationary process $X(t)$ with spectral measure μ and gives a spectral characterization of deterministic and nondeterministic processes.

Let $X(t)$ be a non-degenerate stationary process with spectral measure μ of the form $d\mu(\lambda) = f(\lambda)d\lambda + d\mu_s(\lambda)$.

(A) *(Kolmogorov-Szegő Theorem).*

$$\lim_{n\to\infty} \sigma_n^2(\mu) = \lim_{n\to\infty} \sigma_n^2(f) = \sigma^2(f) = 2\pi G(f), \tag{3.1}$$

where $G(f)$ is the geometric mean of f, namely

$$G(f) := \begin{cases} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda\right\} & \text{if } \ln f \in L^1(\Lambda) \\ 0, & \text{otherwise,} \end{cases} \tag{3.2}$$

- *It is remarkable that the limit in (3.1) is independent of μ_s.*

Kolmogorov-Szegő Theorem

(B) *(Kolmogorov-Szegő alternative). Either*

$$\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda = -\infty \Leftrightarrow \sigma^2(f) = 0 \Leftrightarrow X(t) \textit{ is deterministic},$$

or else

$$\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda > -\infty \Leftrightarrow \sigma^2(f) > 0 \Leftrightarrow X(t) \textit{ is nondeterministic}.$$

(C) *$X(t)$ is regular (PND) $\Leftrightarrow$ it is nondeterministic and $\mu_s \equiv 0$.*

- The condition $\ln f \in L^1(\Lambda)$ is equivalent to the *Szegő condition*:

$$\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda > -\infty \tag{3.3}$$

(this equivalence follows because $\ln f(\lambda) \le f(\lambda)$).

Formulas for the prediction error

Formulas for the prediction error

We present formulas for the finite prediction error $\sigma_n^2(\mu)$.

- Using Kolmogorov's isometric isomorphism $V : X(t) \leftrightarrow e^{it}$, for $\sigma_n^2(\mu)$ we have

$$\sigma_n^2(\mu) := \min_{\{c_k\}} \mathbb{E} \left| X(0) - \sum_{k=1}^{n} c_k X(-k) \right|^2 = \min_{\{q_n \in \mathcal{Q}_n\}} \|q_n\|_{2,\mu}^2 , \quad (4.1)$$

  where $|| \cdot ||_{2,\mu}$ is the norm in $L^2(\mathbb{T}, \mu)$, and

$$\mathcal{Q}_n := \left\{ q_n : q_n(z) = z^n + c_1 z^{n-1} + \cdots c_n \right\} \quad (4.2)$$

  is the class of monic polynomials (i.e. with $c_0 = 1$) of degree n.
- Thus, the problem of finding $\sigma_n^2(\mu)$ becomes to the problem of finding the solution of the minimum problem (4.1)-(4.2).

Formulas for the prediction error

- The polynomial $p_n(z) := p_n(z,\mu)$ which solves the minimum problem (4.1)-(4.2) is called the *optimal polynomial* for μ in the class $\mathcal{Q}_n$.
- The next result by Szegő solves the minimum problem (4.1)-(4.2).

Proposition (3.1. Szegő)

The unique solution of the minimum problem (4.1)-(4.2) *is given by* $p_n(z) = \kappa_n^{-1}\varphi_n(z)$*, and the minimum in* (4.1) *is equal to* $\|p_n\|^2_{2,\mu} = \kappa_n^{-2}$*, where* $\varphi_n(z) = \kappa_n z^n + \cdots + l_n$ $(\kappa_n > 0)$ *is the* n^{th} *orthogonal polynomial on the unit circle associated with the measure* μ*.*

- Thus, for the prediction error $\sigma_n^2(\mu)$ we have the following formula:

$$\sigma_n^2(\mu) = \min_{\{q_n \in \mathcal{Q}_n\}} \|q_n\|^2_{2,\mu} = \|p_n(\mu)\|^2_{2,\mu} = \left\|\kappa_n^{-1}\varphi_n(\mu)\right\|^2_{2,\mu} = \kappa_n^{-2}. \quad (4.3)$$

- In the theory of OPUC and prediction theory an important role play the following numbers, called the *parameters* or *Verblunsky coefficients*:

$$v_n := v_n(\mu) = -\overline{p_n(0)} = -\kappa_n^{-1}\varphi_n(0) = \bar{l}_n\kappa_n^{-1}, \quad |v_n| < 1, n \in \mathbb{N}. \tag{4.4}$$

There is a close relationship between the prediction error $\sigma_n^2(\mu)$ and the parameters v_n, given by formulas:

$$\sigma_n^2(\mu) = \prod_{j=1}^{n}(1 - |v_j|^2) \quad \text{and} \quad \frac{\sigma_{n+1}^2(\mu)}{\sigma_n^2(\mu)} = 1 - |v_n|^2. \tag{4.5}$$

From the second formula in (4.5), it follows that the convergence of the sequences $|v_n|$ and $\sigma_{n+1}(\mu)/\sigma_n(\mu)$ are equivalent.

Formulas for the prediction error

- For a general measure μ the asymptotic relation

$$\lim_{n\to\infty} v_n(\mu) = 0 \tag{4.6}$$

  is of special interest.
- In this respect the following question arises naturally:
- what is the "minimal" sufficient condition on μ ensuring (4.6)?
- The next result of Rakhmanov (1983) shows that for (4.6), or equivalently, for

$$\lim_{n\to\infty} \sigma_{n+1}(\mu)/\sigma_n(\mu) = 1$$

  it is enough only to have a.e. positiveness on $\mathbb{T}$ of the s.d. f.

Theorem (Rakhmanov)

Let the measure μ have the form: $d\mu(\lambda) = f(\lambda)d\lambda + d\mu_s(\lambda)$, with $f > 0$ a.e. on $\mathbb{T}$. Then the asymptotic relation (4.6) is satisfied.

Asymptotic behavior of the prediction error for singular processes.
Background:
Rosenblatt's and Davisson's results.

Asymptotic for singular processes

- Only few works are devoted to the study of the speed of convergence of $\delta_n(f) = \sigma_n^2(f)$ to zero as $n \to \infty$, that is, the asymptotic behavior of the prediction error for deterministic processes.
- Using the technique of OPUC, M. Rosenblatt (1957) investigated the asymptotic behavior of the prediction error $\sigma_n^2(f)$ for deterministic processes in the following two cases:
  - (A) the spectral density $f(\lambda)$ is continuous and positive on a segment of $[-\pi, \pi]$ and zero elsewhere.
  - (B) the spectral density $f(\lambda)$ has a very high order of contact with zero at points $\lambda = 0, \pm\pi$, and is strictly positive otherwise.
- We will say that the spectral density $f(\lambda)$ has a *very high order of contact with zero at a point* λ_0 if $f(\lambda)$ is positive everywhere except for the point λ_0, due to which the Szegő condition (3.3) is violated.

Rosenblatt's first theorem about speed of convergence of $\sigma_n^2(f)$.

- For the case (a) above, M. Rosenblatt proved that the prediction error $\sigma_n^2(f)$ decreases to zero exponentially as $n \to \infty$. More precisely, M. Rosenblatt proved the following theorem.

Theorem (Rosenblatt's first theorem (Theorem 1))

Let the s.d. f of a stationary process $X(t)$ be positive and continuous on the segment $[\pi/2 - \alpha, \pi/2 + \alpha]$, $0 < \alpha < \pi$, and zero elsewhere. Then $\sigma_n^2(f)$ approaches zero exponentially as $n \to \infty$. More precisely,

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = \sin(\alpha/2). \tag{5.1}$$

Davisson's theorem.

- Using constructive methods, Davisson (1965) obtained an upper bound (rather than an asymptote) for the prediction error $\sigma_n^2(f)$ without imposing continuity requirement on the s.d. $f(\lambda)$. Specifically, in Davisson (1965) was proved the following result:

THEOREM (DAVISSON (THEOREM 2))

Let the s.d. $f(\lambda)$, $\lambda \in [-\pi, \pi]$ of the process $X(t)$ be identically zero on a closed interval of length $2\pi - 2\alpha$, $0 < \alpha < \pi$. Then for the prediction error $\sigma_n^2(f)$ the following inequality holds:

$$\sigma_n^2(f) \leq 4c\,(\sin(\alpha/2))^{2n-2}\,,$$

where $c = r(0)$ and $r(\cdot)$ is the covariance function of $X(t)$.

Rosenblatt's second theorem about speed of convergence of $\sigma_n^2(f)$.

- Concerning the case (b), for a specific singular process $X(t)$ Rosenblatt proved that $\sigma_n^2(f)$ decreases to zero *like a power*.
- More precisely, the deterministic process $X(t)$ considered in Rosenblatt (1957) has the spectral density

$$f_a(\lambda) := \frac{e^{(2\lambda-\pi)\varphi(\lambda)}}{\cosh\left(\pi\varphi(\lambda)\right)}, \quad f_a(-\lambda) = f_a(\lambda), \quad 0 \le \lambda \le \pi, \qquad (5.2)$$

where $\varphi(\lambda) = (a/2)\cot\lambda$ and a is a positive parameter.
- For this case, Rosenblatt proved the following theorem.

Theorem (Rosenblatt's second theorem (Theorem 3))

Suppose that the process $X(t)$ has s.d. f_a given by (5.2). Then

$$\sigma_n^2(f_a) \sim \frac{\Gamma^2\left((a+1)/2\right)}{\pi 2^{2-a}}\, n^{-a} \quad \text{as} \quad n \to \infty. \qquad (5.3)$$

- Note that the function in (5.2) was first considered by Pollaczek (1929), and then by Szegő (1935), as a weight-function of a class of orthogonal polynomials possessing certain 'irregular' properties.
- It is worth to note that in Rosenblatt (1957) it was observed the singularity of function $f_a(\lambda)$ only at point $\lambda = 0$, while a detailed analysis showed that for f_a we have the asymptotic relation:

$$f_a(\lambda) \sim \begin{cases} 2e^a \exp\{-a\pi/|\lambda|\} & \text{as } \lambda \to 0, \\ 2\exp\{-a\pi/(\pi - |\lambda|)\} & \text{as } \lambda \to \pm\pi. \end{cases} \tag{5.4}$$

Thus, f_a has very high order of contact with zero at $\lambda = 0, \pm\pi$, due to which the process with s.d. f_a is singular and $\sigma_n^2(f_a)$ decreases to zero like n^{-a}.

Asymptotic for singular processes

- **Remark.** Under the conditions of Rosenblatt's first theorem (Theorem 5.1), we have

$$\lim_{n\to\infty} \sigma^2_{n+1}(f)/\sigma^2_n(f) = \sin^2(\alpha/2) \quad \text{and} \quad \lim_{n\to\infty} |v_n(f)| = \cos(\alpha/2).$$

- Similarly, under the conditions of Rosenblatt's second theorem (Theorem 5.3), we have

$$\lim_{n\to\infty} \sigma_{n+1}(f_a)/\sigma_n(f_a) = 1 \quad \text{and} \quad \lim_{n\to\infty} v_n(f_a) = 0,$$

where $v_n(f)$ and $v_n(f_a)$ are the Verblunsky coefficients corresponding to functions f and f_a, respectively.

- In the rest of this talk we present extensions of the above stated Rosenblatt's theorems (Theorems 1 and 3) and Davisson's theorem (Theorem 2) to broader classes of spectral densities.

Extensions of Rosenblatt's and Davisson's results.

Extensions of Rosenblatt's first theorem.

- In what follows, by E_f we denote the spectrum of the process $X(t)$:

$$E_f := \{e^{i\lambda} : f(\lambda) > 0\}. \tag{5.5}$$

  Thus, the closure $\overline{E}_f$ of E_f is the support of the s.d. f.

- For a compact set F in the complex plane $\mathbb{C}$ by $\tau(F)$ we denote the *transfinite diameter* of F.

- **Transfinite diameter.** Let F be a compact set in the complex plane $\mathbb{C}$. Given any natural number $n \geq 2$, choose n points $z_1, \ldots, z_n \in F$ so as to maximize the product of the distances between them. Then the geometric mean of these distances, denoted by $\tau_n(F)$, is called the nth *transfinite diameter* of F. Fekete (1930) proved that the sequence $\tau_n(F)$ has a finite limit as $n \to \infty$. This limit, denoted by $\tau(F)$, is called the *transfinite diameter* of F.
  If F is empty or consists of a finite number of points, then $\tau(F) = 0$.

- The next result extends Rosenblatt's first theorem (Theorem 1) to the case of several arcs, without having to stipulate continuity of s.d. f.

THEOREM (THEOREM 4)

Let the support $\overline{E}_f$ of the spectral density f of the process $X(t)$ consist of a finite number of closed arcs of the unit circle $\mathbb{T}$, and let $f > 0$ a.e. on $\overline{E}_f$. Then the sequence $\sqrt[n]{\sigma_n(f)}$ converges, and

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = \tau_f, \tag{5.6}$$

where $\tau_f := \tau(\overline{E}_f)$ is the transfinite diameter of $\overline{E}_f$.

Extensions of Rosenblatt's first theorem

- **Remark.** In Theorem 1, $\overline{E}_f = \{e^{i\lambda} : \lambda \in [\pi/2 - \alpha, \pi/2 + \alpha]\}$, which represents a closed arc of length 2α, and we have $\tau(\overline{E}_f) = \sin(\alpha/2)$. Thus, the asymptotic relation (5.1) is a special case of (5.6).
- **Remark.** It follows from (5.6) that the question of exponential decay of $\sigma_n(f)$ is determined solely by the transfinite diameter of the support $\overline{E}_f$ of the s.d. f, and does not depend on the values of f on $\overline{E}_f$.
- The following result provides a sufficient condition for the exponential decay of $\sigma_n(f)$.

Theorem (Theorem 5)

If the spectral density f of the process $X(t)$ vanishes on an arc, then the prediction error $\sigma_n(f)$ decreases to zero exponentially. More precisely, if f vanishes on an arc $\Gamma_\delta \subset \mathbb{T}$ of length 2δ $(0 < \delta < \pi)$, then

$$\limsup_{n\to\infty} \sqrt[n]{\sigma_n(f)} \leqslant \cos(\delta/2) < 1. \qquad (5.7)$$

- The next result gives a necessary condition for the exponential decay of $\sigma_n(f)$.

THEOREM (THEOREM 6)

A necessary condition for $\sigma_n(f)$ to tend to zero exponentially is that the s.d. f should vanish on a set of positive Lebesgue measure.

- **Remark.** This theorem shows that if the s.d. f a.e. positive, then it is impossible to obtain exponential decay of the prediction error $\sigma_n(f)$, no matter how high the orders of the zeros of f.

- From Theorem 4, we obtain the following result.

Theorem (Theorem 7)

Let the support $\overline{E}_f$ and the s.d. f satisfy the conditions of Theorem 4. If the sequence of Verblunsky coefficients $v_n(f)$ converges in modulus, then

$$\lim_{n\to\infty} |v_n(f)| = \sqrt{1-\tau_f^2}. \tag{5.8}$$

- **Remark.** observe that the convergence of $|v_n(f)|$ (or equivalently $\sigma_{n+1}(f)/\sigma_n(f)$) implies the convergence of $\sqrt[n]{\sigma_n(f)}$, but not the converse. Hence, the condition of convergence (in modulus) of Verblunsky sequence $v_n(f)$ in Theorem 7 is essential.

- From Theorem 4 we obtain the following result, which is a partial converse of Rakhmanov's theorem:

Theorem (Theorem 8)

If the sequence $\sigma_n(f)$ satisfies the following condition:

$$\limsup_{n\to\infty} \sqrt[n]{\sigma_n(f)} = 1 \tag{5.9}$$

(in particular, if $\lim_{n\to\infty} v_n(f) = 0$), then $\overline{E}_f = \mathbb{T}$, i.e. the spectrum of the process is dense in $\mathbb{T}$.

Examples.

- The calculation of the transfinite diameter is a challenging problem, and in only very few cases has the transfinite diameter been exactly calculated.
- Below we give some examples, where we can explicitly calculate the the transfinite diameter by using some properties of the transfinite diameter.
- In examples below we will use the following notation: given $0 < \beta < 2\pi$ and $z_0 = e^{i\theta_0}$, $\theta_0 \in [-\pi, \pi)$, we denote by $\Gamma_\beta(\theta_0)$ an arc of the unit circle of length β which is symmetric with respect to the point $z_0 = e^{i\theta_0}$:

$$\Gamma_\beta(\theta_0) := \{e^{i\theta} : \theta \in [\theta_0 - \beta/2, \theta_0 + \beta/2]\}. \tag{5.10}$$

Examples. A consequence of Theorem 4

Proposition

The transfinite diameter possesses the following properties.

(A) *For a compact set $F \subset \mathbb{C}$ the transfinite diameter $\tau(F)$ is invariant with respect to parallel translation and rotation of F.*

(B) *The transfinite diameter of an arc Γ_α of a circle of radius R with central angle α is equal to $R \sin(\alpha/4)$.*

(C) *The transfinite diameter of an arbitrary line segment F is equal to one-fourth its length, that is, if $F := [a, b]$, then $\tau(F) = \tau([a, b]) = (b - a)/4$.*

(D) *Let $F \subset \mathbb{C}$ be a compact set lying on the unit circle $\mathbb{T}$ and symmetric with respect to real axis, and let F^x be the projection of F onto the real axis. Then $\tau(F) = [2\tau(F^x)]^{1/2}$.*

- **Example 1.** Let $\Gamma_{2\alpha} := \Gamma_{2\alpha}(0)$. Then the projection $\Gamma^x_{2\alpha}$ of $\Gamma_{2\alpha}$ onto the real axis is the segment $[\cos\alpha, 1]$ (see Figure 1a)), and by Proposition (C) for the transfinite diameter $\tau(\Gamma^x_{2\alpha})$ we have

$$\tau(\Gamma^x_{2\alpha}) = (1-\cos\alpha)/4 = (1/2)\sin^2(\alpha/2).$$

Hence, according to Proposition (D), we obtain

$$\tau(\Gamma_{2\alpha}) = [2\tau(\Gamma^x_{2\alpha})]^{1/2} = \sin(\alpha/2). \tag{5.11}$$

Taking into account that the transfinite diameter is invariant with respect to rotation (see Proposition (A)), from (5.11) for any $\theta_0 \in [-\pi, \pi)$ we have $\tau(\Gamma_{2\alpha}(\theta_0)) = \sin(\alpha/2)$.

- **Remark.** Notice that the expression $\sin(\alpha/2)$ in (5.11) was first obtained by Szegő (1935), where he calculated it as the Chebyshev constant of the arc $\Gamma_{2\alpha}(\pi/2)$, then it was deduced by Rosenblatt (1957), as the capacity of $\Gamma_{2\alpha}(\pi/2)$.

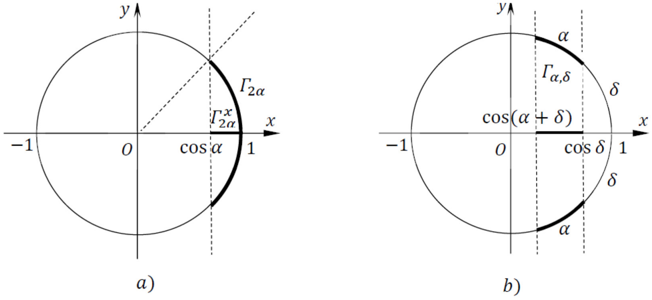

Figure: a) The sets $\Gamma_{2\alpha}$ and $\Gamma_{2\alpha}^{x}$. b) The set $\Gamma_{\alpha,\delta}$.

- **Example 2.** Let $\alpha > 0$, $\delta \geq 0$ and $\alpha + \delta \leq \pi$. Define (Figure 1b)):

$$\Gamma_{\alpha,\delta} := \Gamma_{\alpha,\delta}(0) = \{e^{i\theta} : \theta \in [-(\delta+\alpha), -\delta] \cup [\delta, \delta+\alpha]\}. \quad (5.12)$$

- Then the projection $\Gamma^x_{\alpha,\delta}$ of $\Gamma_{\alpha,\delta}$ onto the real axis is the segment $\Gamma^x_{\alpha,\delta} = [\cos(\alpha+\delta), \cos\delta]$, and by Proposition (C) we have

$$\tau(\Gamma^x_{\alpha,\delta}) = \frac{\cos\delta - \cos(\alpha+\delta)}{4} = \frac{\sin(\alpha/2)\sin(\alpha/2+\delta)}{2}.$$

Hence, according to Proposition (D), we obtain

$$\tau(\Gamma_{\alpha,\delta}) = [2\tau(\Gamma^x_{\alpha,\delta})]^{1/2} = (\sin(\alpha/2)\sin(\alpha/2+\delta))^{1/2}. \quad (5.13)$$

- By Proposition (A), from (5.13) for any $\theta_0 \in [-\pi, \pi)$ we have

$$\tau(\Gamma_{\alpha,\delta}(\theta_0)) = (\sin(\alpha/2)\sin(\alpha/2+\delta))^{1/2}. \quad (5.14)$$

- Observe that for $\delta = 0$ we have $\Gamma_{\alpha,\delta}(\theta_0) = \Gamma_{2\alpha}(\theta_0)$, and formula (5.14) becomes (5.11).

Examples. A consequence of Theorem 4

- Now we apply Theorem 4 to obtain the asymptotic behavior of $\sigma_n(f)$ in the cases where the spectrum of $X(t)$ is as in Examples 1 and 2.

Theorem (Theorem 9)

Let $\overline{E}_f$ be the support of the s.d. f of a stationary process $X(t)$, and let $f > 0$ a.e. on $\overline{E}_f$. Then for $\sigma_n(f)$ the following assertions hold.

(a) *If $\overline{E}_f = \Gamma_{2\alpha}(\theta_0)$, where $\Gamma_{2\alpha}(\theta_0)$ is as in Example 1, then*

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = \sin(\alpha/2).$$

(b) *If $\overline{E}_f = \Gamma_{\alpha,\delta}(\theta_0)$, where $\Gamma_{\alpha,\delta}(\theta_0)$ is as in Example 2, then*

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = \left(\sin(\alpha/2)\sin(\alpha/2+\delta)\right)^{1/2}.$$

An extension of Davisson's theorem

- Recall Davisson's theorem: *If the s.d. f is identically zero on a closed interval of length $2\pi - 2\alpha$, $0 < \alpha < \pi$, then*

$$\sigma_n^2(f) \leq 4c\,(\sin(\alpha/2))^{2n-2}\,, \tag{5.15}$$

  where $c = r(0)$ and $r(\cdot)$ is the covariance function of $X(t)$.

- The theorem that follows extends Davisson's theorem to the case where the spectrum of $X(t)$ consists of a union of two equal arcs.
- Let $\alpha > 0$, $\delta \geq 0$ and $\alpha + \delta \leq \pi$, and let

$$\Gamma_{\alpha,\delta} := \Gamma_{\alpha,\delta}(0) = \{e^{i\theta} : \theta \in [-(\delta+\alpha), -\delta] \cup [\delta, \delta+\alpha]\},$$

  which is the union of two arcs of the unit circle of lengths α, the distance between which (over the circle) is equal to 2δ (see Example 2 and Figure 1b)).

THEOREM (THEOREM 10)

Let the s.d. $f(\lambda)$, $\lambda \in [-\pi, \pi]$ *of the process* $X(t)$ *be nonnegative on the set* $\Gamma_{\alpha,\delta}$ $(\alpha > 0, \delta \geq 0, \alpha + \delta \leq \pi)$ *and vanishes outside* $\Gamma_{\alpha,\delta}$. *Then*

$$\sigma_n^2(f) \leq 4c\,(\sin(\alpha/2))^{n-1}\,(\sin(\alpha/2+\delta))^{n-1}\,, \qquad (5.16)$$

where $c = r(0)$ *and* $r(\cdot)$ *is the covariance function of* $X(t)$.

- **Remark.** For $\delta = 0$ the set $\Gamma_{\alpha,\delta}$ defined by (5.12) is an arc of length 2α, and, in this case, the inequality (5.16) becomes Davisson's inequality (5.15).

Extensions of Rosenblatt's second theorem

- Here we analyze the asymptotic behavior of $\sigma_n^2(f)$ in the case where the s.d. f has a very high order contact with zero at one or several points, so that the Szegő condition (3.3) is violated.
- The approach is based on the asymptotic behavior of the ratio:

$$\frac{\sigma_n^2(fg)}{\sigma_n^2(f)} \quad \text{as } n \to \infty,$$

where g is a non-negative function.

- To clarify the approach, we first assume that f is the s.d. of a nondeterministic process, in which case the geometric mean $G(f)$ is positive. We can then write

$$\lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = \frac{\sigma_\infty^2(fg)}{\sigma_\infty^2(f)} = \frac{2\pi G(fg)}{2\pi G(f)} = \frac{G(f)G(g)}{G(f)} = G(g). \quad (5.17)$$

- It turns out that under some additional assumptions imposed on functions f and g, the asymptotic relation (5.17) remains also valid in the case of deterministic processes, that is, when $G(f) = 0$.

Extensions of Rosenblatt's second theorem

Preliminaries.

- In what follows we consider the class of *singular* processes with s.d. f for which the sequence of $\{\sigma_n(f)\}$ is *weakly varying*, that is,

$$\lim_{n\to\infty} \sigma_{n+1}(f)/\sigma_n(f) = 1.$$

  Denote by $\mathcal{F}$ the class of the corresponding spectral densities:

$$\mathcal{F} := \left\{ f \in L^1(\Lambda) : \; f \geq 0, \; G(f) = 0, \; \lim_{n\to\infty} \frac{\sigma_{n+1}(f)}{\sigma_n(f)} = 1 \right\}. \quad (5.18)$$

- **Remark**. According to Rakhmanov's theorem, a sufficient condition for $f \in \mathcal{F}$ is that $f > 0$ almost everywhere on Λ and $G(f) = 0$. On the other hand, the class $\mathcal{F}$ does not contain spectral densities, which vanish on an entire segment of Λ (or on an arc of $\mathbb{T}$).

- **Def.** Let $\mathcal{F}$ be the class of spectral densities defined by (5.18). For $f \in \mathcal{F}$ denote by $\mathcal{M}_f$ the class of nonnegative functions $g(\lambda)$ $(\lambda \in \Lambda)$ satisfying the conditions: $G(g) > 0$, $fg \in L^1(\Lambda)$, and

$$\lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g), \tag{5.19}$$

that is,

$$\mathcal{M}_f := \left\{ g \geq 0,\ G(g) > 0,\ fg \in L^1(\Lambda),\ \lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g) \right\}.$$

- **Def.** We define the class B to be the set of all nonnegative, Riemann integrable on $\Lambda = [-\pi, \pi]$ functions $h(\lambda)$. Also, define the subclasses:

$$B_+ = \{h \in B : h(\lambda) \geqslant m\},\ B^- = \{h \in B : h(\lambda) \leqslant M\},\ B_+^- = B_+ \cap B^-,$$

where m and M are some positive constants.

- The next theorem describes the asymptotic behavior of the ratio $\sigma_n^2(fg)/\sigma_n^2(f)$ as $n \to \infty$, and states that if the s.d. $f \in \mathcal{F}$, and g is a nonnegative function, which can have *polynomial* type singularities, then $\{\sigma_n(fg)\}$ and $\{\sigma_n(f)\}$ have the same asymptotic behavior.

THEOREM (THEOREM 11)

Let $f \in \mathcal{F}$, and let g be a function of the form:

$$g(\lambda) = h(\lambda) \cdot \frac{t_1(\lambda)}{t_2(\lambda)}, \quad \lambda \in \Lambda, \tag{5.20}$$

where $h \in B_+^-$, t_1 and t_2 are nonnegative trigonometric polynomials, such that $fg \in L^1(\Lambda)$. Then $g \in \mathcal{M}_f$ and $fg \in \mathcal{F}$, that is, fg is the s.d. of a singular process with weakly varying prediction error, and (5.19) holds.

- The next theorem extends Theorem 11 to a broader class of spectral densities, for which the function g can have *arbitrary power type singularities*.

Theorem (Theorem 12)

Let $f \in \mathcal{F}$, and let g be a function of the form:

$$g(\lambda) = h(\lambda) \cdot |t(\lambda)|^{\alpha}, \quad \alpha > 0, \ \lambda \in \Lambda, \tag{5.21}$$

where $h \in B_{+}^{-}$ and t is an arbitrary trigonometric polynomial. Then $g \in \mathcal{M}_f$ and $fg \in \mathcal{F}$, that is, fg is the s.d. of a deterministic process with weakly varying prediction error, and the relation (5.19) holds.

-

- Taking into account that the sequence $\{n^{-\alpha},\ n \in \mathbb{N}, \alpha > 0\}$ is weakly varying, as an immediate consequence of Theorems 11 and 12, we have the following result.

COROLLARY

Let the functions f and g satisfy the conditions of one of Theorems 11 and 12, and let $\sigma_n(f) \sim cn^{-\alpha}$ $(c > 0, \alpha > 0)$ as $n \to \infty$. Then

$$\sigma_n(fg) \sim cG(g)n^{-\alpha} \quad \text{as} \quad n \to \infty,$$

where $G(g)$ is the geometric mean of g.

- The next result, which immediately follows from Theorem 3 and Corollary, extends Rosenblatt's second theorem.

Extensions of Rosenblatt's second theorem

Theorem (Theorem 13)

Let $f = f_a g$, where f_a is defined by (5.2), and let g be a function satisfying the conditions of one of Theorems 11 and 12. Then

$$\delta_n(f) = \sigma_n^2(f) \sim \frac{\Gamma^2\left(\frac{a+1}{2}\right) G(g)}{\pi 2^{2-a}} \, n^{-a} \quad \text{as} \quad n \to \infty,$$

where $G(g)$ is the geometric mean of g.

- We thus have the same limiting behavior for $\sigma_n^2(f)$ as in the Rosenblatt's relation (5.3) up to an additional positive factor $G(g)$.
- In view of Rakhmanov's theorem, Theorems 11-13 remain true if the condition $f \in \mathcal{F}$ is replaced by the slightly strong but more constructive condition: '$f > 0$ *a.e. on* Λ *and* $G(f) = 0$'.

Examples.

- **Example 1.** Let $g(\lambda) = \sin^{-2k}(\lambda - \lambda_0)$, where $k \in \mathbb{N}$ and λ_0 is an arbitrary point from $[-\pi, \pi]$. Then, for the geometric mean $G(g)$ we have $G(g) = 4^k$, and in view of (5.19), we get

$$\lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g) = 4^k.$$

- Thus, dividing the spectral density f by the non-negative trigonometric polynomial $\sin^{2k}(\lambda - \lambda_0)$ of degree $2k$ ($k \in \mathbb{N}$), yields a 4^k-fold asymptotic increase of the prediction error.
- Notice that the value of the geometric mean $G(g)$ does not depend on the choice of the point $\lambda_0 \in [-\pi, \pi]$.

EXAMPLES

- **Example 2.** Let $g(\lambda) = |\sin(\lambda - \lambda_0)|^\alpha$, $\alpha \in \mathbb{R}$. We have

$$\lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g) = \frac{1}{2^\alpha}.$$

- Thus, multiplying the spectral density $f(\lambda)$ by the function $g(\lambda) = |\sin(\lambda - \lambda_0)|^\alpha$ yields a 2^α-fold asymptotic reduction of the prediction error.
- **Example 3.** Let $g(\lambda) = |\lambda - \lambda_0|^\alpha$, $\lambda_0 \in [-\pi, \pi]$, $\alpha \in \mathbb{R}$. We have

$$\lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g) = \left(\frac{\pi}{e}\right)^\alpha \approx (1.156)^\alpha.$$

- Thus, multiplying the spectral density $f(\lambda)$ by the function $g(\lambda) = |\lambda - \lambda_0|^\alpha$ multiplies the prediction error asymptotically by $(\pi/e)^\alpha \approx (1.156)^\alpha$.

An Application. Estimates for the minimal eigenvalue of truncated Toeplitz matrices

Estimates for the minimal eigenvalue

- Here we analyze the relationship between the the minimal eigenvalue of a truncated Toeplitz matrix and the finite prediction error for a stationary process, by showing how it is possible to obtain information about the minimal eigenvalue from that of the prediction error.
- Let $f(\lambda)$ be the s.d. and let

$$T_n(f) := ||r_{k-j}||_{j,k=0,1,\ldots,n}$$

  be the truncated Toeplitz matrix generated by the Fourier coefficients (covariances) of f, and let

$$\lambda_{1,n}(f) \leq \lambda_{2,n}(f) \leq \cdots \lambda_{n+1,n}(f)$$

  be the eigenvalues of $T_n(f)$.
- The next proposition gives a relationship between $\lambda_{1,n}(f)$ and $\sigma_n^2(f)$.

PROPOSITION

Let $\lambda_{1,n}(f)$ and $\sigma_n^2(f)$ be as above. Then $\lambda_{1,n}(f) \le \sigma_n^2(f)$ for any $n \in \mathbb{N}$.

- Applying Theorem 9 and Proposition we obtain asymptotic estimates for $\lambda_{1,n}(f)$.

THEOREM

Let f, $\overline{E}_f$ and $\lambda_{1,n}(f)$ be as above. Then

(a) *If $\overline{E}_f = \Gamma_{2\alpha}(\theta_0)$, where $\Gamma_{2\alpha}(\theta_0)$ is as in Example 1, then*

$$\lambda_{1,n}(f) = O\left(\sin^{2n}(\alpha/2)\right) \quad \text{as} \quad n \to \infty. \tag{6.1}$$

(b) *If $\overline{E}_f = \Gamma_{\alpha,\delta}(\theta_0)$, where $\Gamma_{\alpha,\delta}(\theta_0)$ is as in Example 2, then*

$$\lambda_{1,n}(f) = O\left((\sin(\alpha/2)\sin(\alpha/2+\delta))^n\right) \quad \text{as} \quad n \to \infty. \tag{6.2}$$

- Using Davisson's theorem (Theorem 2), its extension (Theorem 10) and Proposition we obtain exact upper bounds for $\lambda_{1,n}(f)$ rather than the asymptotic estimates (6.1) and (6.2).

THEOREM

Let f, $\overline{E}_f$ and $\lambda_{1,n}(f)$ be as above. Then

(a) *If $\overline{E}_f = \Gamma_{2\alpha}(\theta_0)$, where $\Gamma_{2\alpha}(\theta_0)$ is as in Example 1, then*

$$\lambda_{1,n}(f) \le 4c\,(\sin(\alpha/2))^{2n-2}. \tag{6.3}$$

(b) *If $\overline{E}_f = \Gamma_{\alpha,\delta}(\theta_0)$, where $\Gamma_{\alpha,\delta}(\theta_0)$ is as in Example 2, then*

$$\lambda_{1,n}(f) \le 4c\,(\sin(\alpha/2))^{n-1}\,(\sin(\alpha/2+\delta))^{n-1}, \tag{6.4}$$

where the constant c is as in Davisson's theorem: $c = r(0) = \int_{-\pi}^{\pi} f(\lambda)d\lambda$.

Thank you

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